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CLASSIFICATION OF SIMPLY CONNECTED FOUR-DIMENSIONAL RR-MANIFOLDS

BY

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ABSTRACT. Let (M, g) be a Riemannian manifold. We assume that there is a mapping $s: M \longrightarrow I(M)$, where $I(M)$ is the group of isometries of (M, g), such that $s_x = s(x)$, $\forall x \in M$, has x as a fixed isolated point, then (M, g) is called a Riemannian s-manifold. If the tensor field S on M defined by the relation $S_x = (ds_x)_x$, $\forall x \in M$, is differentiable and invariant by each isometry s_x, then the manifold (M, g) is called a regularly s-symmetric Riemannian manifold. The aim of the present paper is to classify simply connected four-dimensional regularly s-symmetric Riemannian manifolds.

Introduction. Let (M, g) be a connected Riemannian manifold. We assume that there is a family of isometries $\{s_x: x \in M\}$ on (M, g) such that to each point $x \in M$ corresponds an isometry s_x which has this point fixed isolated. The isometry s_x is called Riemannian symmetry at x or simply symmetry at x. The family of Riemannian symmetries $\{s_x: x \in M\}$ is called Riemannian s-structure or simply s-structure on (M, g).

Every Riemannian s-structure $\{s_x: x \in M\}$ on (M, g) determines a tensor field S of type (1, 1) defined by $S_x = (ds_x)_x$. This tensor field is called symmetry tensor field of the $\{s_x: x \in M\}$. If S is smooth then the s-structure $\{s_x: x \in M\}$ is called smooth. If the tensor field S of a smooth s-structure $\{s_x: x \in M\}$ is invariant by each s_x, then it is called regular. A Riemannian manifold (M, g) with a regular s-structure is called regularly s-symmetric Riemannian manifold or briefly RR-manifold and is denoted by (M, g, s). If there is an integer k such that $s_x^k = \mathrm{id}$, $\forall x \in M$, then the RR-manifold is called regularly k-symmetric Riemannian manifold and the positive integer k is called regular order of the s-structure $\{s_x: x \in M\}$.

The aim of the present paper is the classification of all simply connected regularly s-symmetric Riemannian manifolds of four dimensions. There are exactly two categories of such manifolds. The first category contains the Riemannian symmetric spaces of four dimensions, which are known [9, pp. 283-289].

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The second category contains only one *RR*-manifold whose regular order is four, which is given in §7.

2. In this section we give some additional definitions and a few known results.

Let (M, g, s) be an *RR*-manifold with regular *s*-structure $\{s_x: x \in M\}$. It was mentioned above that to this *s*-structure $\{s_x: x \in M\}$ we can associate a tensor field S of type $(1, 1)$. If S is a parallel tensor field, then the *s*-structure $\{s_x: x \in M\}$ is called parallel regular *s*-structure on (M, g).

The following theorem is known [6].

THEOREM (I). *Any regularly s-symmetric Riemannian manifold is regularly k-symmetric for some k, i.e. of finite regular order.*

From the above theorem we conclude that the classification of simply connected four-dimensional regularly *s*-symmetric Riemannian manifolds is reduced to the classification of *RR*-manifolds of finite regular order.

In the classification below we assume the regular order $k \geq 3$ because if $k = 2$, then the *RR*-manifolds are Riemannian symmetric spaces, which have been classified [9, pp. 283–289].

We also assume that the regular *s*-structure on M is not parallel, because if it is parallel then the manifold is a locally symmetric space and since it is simply connected, then M is a symmetric space [7, p. 457].

Let (M, g, s) be an *RR*-manifold with regular *s*-structure $\{s_x: x \in M\}$. The following are known [6].

LEMMA (I). *Let (M, g, s) be an RR-manifold, and let B be an S-invariant tensor field of type* $(1, 2)$ *on (M, g, s). Let $\theta_1, \dots, \theta_n$ be the eigenvalues of S at a point $P \in M$ and $U_1, \dots, U_n$ a corresponding basis of eigenvectors in the complex vector space $M_P^{\mathbf{C}} = M_P \otimes \mathbf{C}$. Then*:

(a) *If $\theta_i\theta_j$ is not an eigenvalue, then $B(U_i, U_j) = 0$.*

(b) *If $\theta_i\theta_j$ is an eigenvalue θ_k, then $B(U_i, U_j)$ belongs to the eigenspace corresponding to θ_k.*

COROLLARY (I). *Let (M, g, s) be an RR-manifold with a nonparallel regular s-structure $\{s_x: x \in M\}$. If $\theta_1, \dots, \theta_n$ are the eigenvalues of S at a point $P \in M$, then in the set of all products $\{\theta_i\theta_j, 1 \leq i < j \leq n\}$ there is at least one eigenvalue.*

For a simply connected four-dimensional homogeneous Riemannian manifold the following theorem is known [4, p. 363].

THEOREM (II). *Let $M = G/H$ be a simply connected four-dimensional homogeneous Riemannian manifold. If* $\dim H > 1$ *then M is a symmetric space.*

For any *RR*-manifold we have the theorem [7, p. 452],

THEOREM (III). *Every RR-manifold* (M, g, s) *is a homogeneous space, i.e.* $M = G/H$.

It is known that $\dim M = \dim G - \dim H$, where G is a transitive Lie group of isometries of M and H the isotropy subgroup of G at a point of M. If $\dim M = 4$, then the maximal dimensions of G and H are ten and six, respectively and their minimal dimensions are four and zero respectively.

In the classification below we shall study only the case $\dim H \leqslant 1$, because if $\dim H \geqslant 2$, the manifold M is symmetric space and the classification of such spaces has been done.

3. Let (M, g, s) be a simply connected four-dimensional *RR*-manifold with nonparallel regular s-structure $\{s_x : x \in M\}$ of order k. The manifold M is a homogeneous space, i.e. $M = G/H$. Let $\overline{\nabla}$ be the canonical connection of the second kind on the homogeneous space $M = G/H$. Let $\overline{R}$, $\overline{T}$ be the curvature and torsion tensor fields of $\overline{\nabla}$, respectively.

PROPOSITION (I). *If S is the symmetry tensor field on* (M, g, s), *then the eigenvalues of S are of the form* $\varphi, \overline{\varphi}, \varphi^2, \overline{\varphi}^2$.

PROOF. Let P be a point of the manifold M. Then S_P is an orthogonal transformation on the tangent space $T_P(M)$. By complexification of $T_P(M)$ we obtain the complex vector space $T_P^{\mathbf{C}}(M) = \mathbf{C} \otimes T_P(M)$ and from the linear transformation S_P we have another linear transformation on $T_P^{\mathbf{C}}(M)$, which is denoted by $S_P^{\mathbf{C}}$.

The dimension of the vector space $T_P^{\mathbf{C}}(M)$ is four over $\mathbf{C}$. Therefore there are four eigenvalues of $S_P^{\mathbf{C}}$, which have the form

$$(3.1)\qquad \begin{aligned} \theta_1 &= \varphi_1 = \cos\frac{2\pi v}{k} + i\sin\frac{2\pi v}{k} = e^{2\pi i v/k},\\ \theta_2 &= \overline{\varphi}_1 = \cos\frac{2 v v}{k} - i\sin\frac{2\pi v}{k} = e^{-2\pi i v/k},\end{aligned}$$

$$(3.2)\qquad \begin{aligned} \theta_3 &= \varphi_2 = \cos\frac{2\pi\lambda}{k} + i\sin\frac{2\pi i\lambda}{k} = e^{2\pi i\lambda/k},\\ \theta_4 &= \overline{\varphi}_2 = \cos\frac{2\pi\lambda}{k} - i\sin\frac{2\pi\lambda}{k} = e^{-2\pi i\lambda/k}.\end{aligned}$$

Since the regular s-structure $\{s_x : x \in M\}$ on *RR*-manifold (M, g, s) is not parallel, we conclude that the set

$$I = \{\theta_i\theta_j : 1 \leqslant i < j \leqslant 4\} = \{\varphi_1\overline{\varphi}_1 = \varphi_2\overline{\varphi}_2 = 1, \varphi_1\varphi_2, \varphi_1\overline{\varphi}_2, \overline{\varphi}_1\varphi_2, \overline{\varphi}_1\overline{\varphi}_2\}$$

contains at least one eigenvalue of $S_P^{\mathbf{C}}$.

In order to define the order of the nonparallel regular s-structure $\{s_x\colon x \in M\}$ we distinguish the following cases:

(i) There are two distinct eigenvalues of $S_p^{\mathbf{C}}$, that is, $\theta_1 = \theta_3 = \varphi_1 = \varphi_2$ and $\theta_2 = \theta_4 = \overline{\varphi}_1 = \overline{\varphi}_2$. Therefore the set I contains the elements $I = \{\varphi_1\overline{\varphi}_1 = \varphi_2\overline{\varphi}_2 = 1, \varphi_1^2, \overline{\varphi}_1^2\}$ which has at least one eigenvalue of $S_p^{\mathbf{C}}$. Therefore we have $\varphi_1^2 = \overline{\varphi}_1$ from which we obtain $\overline{\varphi}_1^2 = \varphi_1, \varphi_1^3 = 1$ and $\overline{\varphi}_1^3 = 1$. Hence the order of $\{s_x\colon x \in M\}$ is three and the conditions of the proposition are satisfied, i.e. $\varphi = \varphi_1, \overline{\varphi} = \overline{\varphi}_1, \varphi^2 = \overline{\varphi}_1, \overline{\varphi}^2 = \varphi_1$.

(ii) We assume that the distinct eigenvalues of $S_p^{\mathbf{C}}$ are three. Therefore the two equal eigenvalues must be -1, i.e. $\varphi_2 = \overline{\varphi}_2 = -1$. In this case the set $I = \{1, -\varphi_1, -\overline{\varphi}_1\}$ must contain at least one eigenvalue of $S_p^{\mathbf{C}}$. Hence we have $-\varphi_1 = \overline{\varphi}_1$ from which we obtain $\varphi_1 = i, \overline{\varphi}_1 = -i$. Therefore if the order of the nonparallel s-structure $\{s_x\colon x \in M\}$ is four, then the eigenvalues of $S_p^{\mathbf{C}}$ are $\varphi_1 = i, \overline{\varphi}_1 = -i, \varphi_2 = \overline{\varphi}_2 = -1$ which satisfy the conditions of the proposition i.e. $\varphi_1 = \varphi = i, \overline{\varphi}_1 = \overline{\varphi} = -i, \varphi_2 = \varphi^2 = -1, \overline{\varphi}_2 = \overline{\varphi}^2 = -1$.

(iii) Finally, we assume that all eigenvalues of $S_p^{\mathbf{C}}$ are distinct. Hence the collection I consists of the following elements $\{\varphi\overline{\varphi}_1 = \varphi_2\overline{\varphi}_2 = 1, \varphi_1\varphi_2, \varphi_1\overline{\varphi}_2, \overline{\varphi}_1\varphi_2, \overline{\varphi}_1\overline{\varphi}_2\}$. Now we suppose that $\varphi_1\overline{\varphi}_2 = \overline{\varphi}_1 \longrightarrow \overline{\varphi}_2 = \overline{\varphi}_1^2$ which, by means of (3.1) and (3.2), takes the form $e^{2\pi i(\nu-\lambda)/k} = e^{-2\pi i\nu/k}$ from which we have

$$\lambda = -kp + 2v, \quad \text{where } p \in Z. \tag{3.3}$$

We can assume that $k > \lambda > v \geqslant 1$ and therefore the eigenvalues of $S_p^{\mathbf{C}}$ are all distinct.

The relations (3.2) by means of (3.3) take the form

$$\begin{aligned} \theta_3 &= \varphi_2 = \cos\frac{4\pi v}{k} + i\sin\frac{4\pi v}{k} = e^{4\pi iv/k}, \\ \theta_4 &= \overline{\varphi}_2 = \cos\frac{4\pi v}{k} - i\sin\frac{4\pi v}{k} = e^{-4\pi iv/k}. \end{aligned} \tag{3.4}$$

If we denote by $\varphi = \varphi_1$, then from (3.1) and (3.4) we obtain that the eigenvalues of $S_p^{\mathbf{C}}$ are of the form $\varphi = \varphi_1, \overline{\varphi} = \overline{\varphi}_1, \varphi^2 = \varphi_2, \overline{\varphi}^2 = \overline{\varphi}_2$. If the order of $\{s_x\}$is equal to five, then the set I contains all the eigenvalues of $S_p^{\mathbf{C}}$, that is $\{\varphi_1\varphi_2 = \overline{\varphi}_2, \varphi_1\overline{\varphi}_2 = \overline{\varphi}_1, \overline{\varphi}_1\varphi_2 = \varphi_1, \varphi_1\overline{\varphi}_2 = \varphi_2\}$.

Finally, if the order of $\{s_x\}$ is greater than five, then the set I contains the following eigenvalues $(\varphi_1\overline{\varphi}_2 = \overline{\varphi}_1, \overline{\varphi}_1\varphi_2 = \overline{\varphi}_1)$ of $S_p^{\mathbf{C}}$.

If we make other assumptions, other than $\varphi_1\overline{\varphi}_2 = \overline{\varphi}_1$, i.e. $\varphi_1\varphi_2 = \overline{\varphi}_1$, we obtain the same results. We always distinguish two cases when the order of $\{s_x\}$ is five and when the order of $\{s_x\}$ is greater than five. Q.E.D.

Therefore the classification of all simply connected four-dimensional RR-manifolds (M, g, s) is reduced to studying the following cases: (i) order$(\{s_x\}) = 3$,

(ii) order $(\{s_x\}) \geqslant 6$, order $(\{s_x\}) = 5$ and finally order $(\{s_x\}) = 4$, which will be studied separately below.

4. We assume that the nonparallel regular s-structure $\{s_x\}$ of the RR-manifold (M, g, s) has order 3. If $P \in M$, then the linear transformation $S_P^{\mathbf{C}}$ on $T_P^{\mathbf{C}}(M)$ has two distinct eigenvalues.

PROPOSITION (II). *If (M, g, s) is a simply connected four-dimensional RR-manifold whose regular s-structure $\{s_x: x \in M\}$ has order* 3, *then the Riemannian manifold (M, g) is a symmetric space.*

PROOF. The RR-manifold (M, g, s) is a homogeneous space $M = G/H$. Let $T_0(M)$ be the tangent space of M at its origin 0. From the symmetry tensor field S we obtain a linear transformation $S_0^{\mathbf{C}}$ on $T_0(M)$ whose two distinct eigenvalues are conjugate complex numbers. We also obtain a linear transformation S_0 on $T_0(M)$. Therefore there exists an orthonormal base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$ such that S_0 can be written in matrix form.

$$S_0 = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} & 0 & 0 \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} & 0 & 0 \\ 0 & 0 & \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ 0 & 0 & \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}.$$

Let $L(H)$ be the Lie algebra of all real endomorphisms A on $T_0(M)$ which, as derivations on the tensor algebra of the vector space $T_0(M)$, satisfy the relations

$$A(S_0) = A(g_0) = A(\overline{T}_0) = A(\overline{R}_0) = 0.$$

Since we have $\overline{\nabla}\overline{T} = \overline{\nabla}\overline{R} = \overline{\nabla}D = \overline{\nabla}S = 0$ where D is the difference tensor field of type (1, 2) on M defined by the relation [1, p. 137]

$$D(Y, X) = (\nabla S)(S^{-1}Y, (I - S)^{-1}X) = (\nabla_{(I-S)^{-1}X}S)(S^{-1}Y),$$

we conclude that $\overline{R}_0(X, Y) \in L(H)$, $\forall X, Y \in T_0(M)$.

From the relation $A(S_0) = 0$ we obtain that the linear transformations A, S_0 commute. Therefore A can be represented by a matrix as follows:

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \beta_1 & \beta_2 \\ -\alpha_2 & \alpha_1 & -\beta_2 & \beta_1 \\ \gamma_1 & \gamma_2 & \delta_1 & \delta_2 \\ -\gamma_2 & \gamma_1 & -\delta_1 & \delta_2 \end{pmatrix}.$$

For the orthonormal base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$ we have $g_0(fi, fi) = 1$, and $g_0(fi, fj) = 0, i \neq j$. From these relations and since $A(g_0) = 0$ we conclude that the matrix A takes the form

$$A = \begin{pmatrix} 0 & a & c & b \\ -a & 0 & -b & c \\ -c & b & 0 & d \\ -b & -c & -d & 0 \end{pmatrix}.$$

If we set

$$V_1 = f_1 + if_2, \qquad V_2 = f_3 + if_4, \tag{4.1}$$

then the vectors $V_1, \bar{V}_1, V_2, \bar{V}_2$ form a base of $T_0^{\mathbb{C}}(M)$.

If we apply Lemma (I) in our case we conclude that

$$\bar{T}_0(V_1, \bar{V}_1) = 0, \qquad \bar{T}_0(V_1, V_2) = \lambda \bar{V}_1 + \mu \bar{V}_2, \tag{4.2}$$

$$\bar{T}_0(V_1, \bar{V}_2) = 0, \qquad \bar{T}_0(V_2, \bar{V}_2) = 0. \tag{4.3}$$

The relations (4.2) and (4.3) by means of (4.1) and after some estimates give

$$\bar{T}_0(f_1, f_2) = 0, \qquad \bar{T}_0(f_3, f_4) = 0, \tag{4.4}$$

$$\bar{T}_0(f_1, f_3) = -\bar{T}_0(f_2, f_4) = \tfrac{1}{2}(\zeta f_1 + \xi f_2 + \sigma \xi_3 + \tau f_4), \tag{4.5}$$

$$\bar{T}_0(f_1, f_4) = \bar{T}_0(f_2, f_3) = \tfrac{1}{2}(\xi f_1 - \zeta f_2 + r f_3 - \sigma f_4). \tag{4.6}$$

From the relations $A(\bar{T}_0) = 0$ and (4.4), (4.5), (4.6) we obtain

$$(2a + d)\zeta = -b\sigma + cr, \qquad (2a + d)\xi = br - c\sigma, \tag{4.7}$$

$$(a + 2d)\sigma = -b\zeta - c\xi, \qquad (a + 2d)r = -b\xi + c\zeta. \tag{4.8}$$

The Lie algebra $L(G)$ of G can be decomposed as follows: $L(G) = L(H) \oplus T_0(M)$ whose Lie bracket is defined by

$$[X, Y] = -\bar{T}_0(X, Y) - \bar{R}_0(X, Y), \qquad X, Y \in T_0(M), \tag{4.9}$$

$$[A, X] = AX, \qquad [A, B] = AB - BA, \qquad A, B \in L(H), X \in T_0(M). \tag{4.10}$$

From the relations (4. 4), (4.5), (4.6), (4.9), (4.10) and from the fact that dim $L(H) \leqslant 1$ we have

$$2[f_1, f_2] = 2\alpha e, \tag{4.11}$$

$$2[f_1, f_3] = -(\zeta f_1 + \xi f_2 + \sigma f_3 + \tau f_4) + 2\beta e, \tag{4.12}$$

$$2[f_1, f_4] = -(\xi f_1 - \zeta f_2 + \tau f_3 - \sigma f_4) + 2\gamma e, \tag{4.13}$$

$$2[f_2, f_3] = -(\xi f_1 - \zeta f_2 + \tau f_3 - \sigma f_4) + 2\delta e, \tag{4.14}$$

$$2[f_2, f_4] = (\zeta f_1 + \xi f_2 + \sigma f_3 + \tau f_4) + 2\vartheta e, \tag{4.15}$$

$$2[f_3, f_4] = 2\upsilon e, \tag{4.16}$$

$$[f_1, e] = af_2 + cf_3 + bf_4, \quad [f_2, e] = -af_1 - bf_3 + cf_4, \tag{4.17}$$

$$[f_3, e] = -cf_1 + bf_2 + df_4, \quad [f_4, e] = -bf_1 - cf_2 - df_4, \tag{4.18}$$

where e is the unit vector of $L(H)$.

The above Lie brackets must satisfy the Jacobi identity. From this we obtain, except the relations (4.7) and (4.8), the following ones

$$-2a\beta + 2c\alpha + \xi\sigma - \zeta\tau = 0, \quad 2a\delta + 2b\alpha + \xi\tau + \zeta\sigma = 0, \tag{4.19}$$

$$b\beta + c\delta = 0, \quad 2b\delta - 2c\beta + 2d\alpha + \sigma^2 + \tau^2 = 0, \tag{4.20}$$

$$-2a\vartheta + 2c\upsilon + \xi\sigma - \zeta\tau = 0, \quad -2a\gamma + 2b\upsilon + \xi\tau + \zeta\sigma = 0, \tag{4.21}$$

$$b\vartheta - c\gamma = 0, \quad -2b\gamma - 2c\vartheta + 2d\upsilon + \sigma^2 + \tau^2 = 0, \tag{4.22}$$

$$2c\upsilon - 2d\beta + \xi\sigma - \zeta\tau = 0, \quad 2b\upsilon - 2d\gamma + \xi\tau + \zeta\sigma = 0, \tag{4.23}$$

$$b\beta - c\gamma = 0, \quad 2a\upsilon - 2b\gamma - 2c\beta + \xi^2 + \zeta^2 = 0, \tag{4.24}$$

$$2c\upsilon - 2d\vartheta + \xi\sigma - \zeta\tau = 0, \quad 2b\upsilon + 2d\delta + \xi\tau + \zeta\sigma = 0, \tag{4.25}$$

$$b\vartheta + c\delta = 0, \quad 2a\upsilon + 2b\beta - 2c\vartheta + \xi^2 + \zeta^2 = 0, \tag{4.26}$$

$$b(\beta + \vartheta) - c(\gamma - \delta) = 0, \tag{4.27}$$

$$a\delta + b(\alpha - \upsilon) + d\gamma = 0, \quad -a\vartheta + c(\alpha - \upsilon) + d\beta = 0, \tag{4.28}$$

$$a\beta + c(\alpha - \upsilon) - d\vartheta = 0, \quad a\gamma - b(\alpha - \upsilon) + d\delta = 0, \tag{4.29}$$

$$\tau(\beta - \vartheta) - \sigma(\gamma + \delta) = 0, \quad \sigma(\beta - \vartheta) + \tau(\gamma + \delta) = 0, \tag{4.30}$$

$$\xi(\beta - \vartheta) - \zeta(\gamma + \delta) = 0, \quad \zeta(\beta - \vartheta) + \xi(\gamma + \delta) = 0. \tag{4.31}$$

Since $\overline{\nabla}$ is the canonical connection of the second kind on M, then we have [5, Vol. II, p. 193]

$$\bar{R}_0(X, Y)Z = -[[X, Y]_b, Z], \quad X, Y, Z \in T_0(M). \tag{4.32}$$

The formula (4.32) for the vectors f_1, f_2, f_3 and by means of (4.11) and the first of (4.18) becomes

$$\bar{R}_0(f_1, f_2)f_3 = \alpha(-cf_1 + bf_2 + df_4);$$

applying the derivation A to this we obtain

$$\bar{R}_0(A(f_1), f_2)f_3 + \bar{R}_0(f_1, A(f_2))f_3 + \bar{R}_0(f_1, f_2)A(f_3) = \alpha(-cA(f_1) + bA(f_2) + dA(f_3))$$

from which by virtue of (4.11)–(4.18) we have

$$\alpha b = 0, \quad b[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \tag{4.33}$$

$$c[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \quad d[b(\beta + \vartheta) - c(\gamma - \delta)] = 0. \tag{4.34}$$

The same formula (4.32) for the vectors f_1, f_2, f_4 and by means of (4.11) and the second of (4.18) takes the form

$$\bar{R}_0(f_1, f_2)f_4 = -\alpha(bf_1 + cf_2 + df_3). \tag{4.35}$$

If we apply the derivation A to (4.35) and after some calculations we obtain

$$\alpha c = 0, \quad b[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \tag{4.36}$$

$$c[b(\beta + \vartheta) - c(\gamma - \delta)] = 0, \quad d[b(\beta + \vartheta) - c(\gamma - \delta)] = 0. \tag{4.37}$$

From the relations (4.7) and (4.8) by virtue of the first of (4.33) and (4.36) and if $\alpha \neq 0$ we conclude that

$$(2a + d)\xi = (2a + d)\zeta = (\alpha + 2d)r = (a + 2d)\sigma = 0. \tag{4.38}$$

If $2a + d \neq 0$ and $a + 2d \neq 0$, then from (4.38) we have $\xi = \zeta = r = \sigma = 0$. If $2a + d = 0$ and $a + 2d \neq 0$, then (4.38) implies $\xi = \zeta = 0$. Therefore the second of (4.24) becomes $2av = 0$, which implies $v = 0$, because if $a = 0$, then $d = 0$ and hence $a + 2d = 0$ which contradicts our assumption $a + 2d \neq 0$.

From the second of (4.20) and (4.22) we conclude that $d(\alpha - v) = 0$ and since $\alpha \neq 0$ and $v = 0$ we have $d = 0$ and from $2a + d = 0$ we take $a = 0$, which contradicts our assumption $2d + a \neq 0$. Therefore the assumption $2a + d = 0$ and $a + 2d \neq 0$ is not valid.

If $2a + d = 0$ and $a + 2d = 0$, then we obtain $a = d = 0$ and from the second of (4.20) and (4.24) we have $\xi = \zeta = r = \sigma = 0$.

Now, we assume that $\alpha = 0$. The systems of equations (4.30) and (4.31) have solutions different from zero, i.e. $r = \sigma = 0$ and $\xi = \zeta = 0$, if the following relations are satisfied:

(4.39) $$\beta - \vartheta = 0, \quad \gamma + \delta = 0.$$

From the assumption $\alpha = 0$, the relations (4.28), (4.29), the second of (4.20) and (4.22) and the relations (4.39) we conclude that $bv = cv = dv = 0$. If $v \neq 0$ we obtain again $b = c = d = 0$ and from this $\xi = \zeta = r = \sigma = 0$.

If $v = 0$ and because $\alpha = 0$, then (4.28) and (4.29) by virtue of (4.39) take the form

$$\beta(a-d) = 0, \quad \gamma(a-d) = 0.$$

If $a \neq 0$, then $\beta = \gamma = \delta = \vartheta = 0$, then from the second of (4.22) and (4.24) we have $\xi = \zeta = r = \sigma = 0$.

If we apply the derivation A to the relations

$$\bar{R}_0(f_1, f_3)f_4 = \beta[f_4, e], \quad \bar{R}_0(f_1, f_3)f_2 = \beta[f_2, e]$$

which are obtained from (4.32) for the pairs $(X = f_1,\ Y = f_2,\ Z = f_4)$ and $(X = f_1,\ Y = f_3,\ Z = f_4)$ respectively, after some calculations we have

(4.41) $$\beta c = 0, \quad b(a\delta - bv + b\alpha + d\vartheta) = 0,$$

(4.42) $$c(a\delta - bv + ba + d\vartheta) = 0, \quad d(a\delta - bv + b\alpha + d\vartheta) = 0,$$

(4.43) $$\beta c = 0, \quad a(a\delta - bv + b\alpha + d\gamma) = 0,$$

(4.44) $$b(a\delta - bv + b\alpha + d\gamma) = 0, \quad c(a\delta - bv + b\alpha + d\gamma) = 0.$$

If $c \neq 0$ and since $\alpha = v = 0$, then from (4.41), (4.42), (4.43) and (4.44) we obtain $\beta = 0$, $a\delta + d\vartheta = 0$ and $a\delta + d\gamma = 0$. We assume that $\gamma \neq 0$, because the case $\gamma = 0$ has been studied. If $\gamma \neq 0$, then we have $a = d = 0$ and hence $\xi = \zeta = r = \sigma = 0$.

If $c \neq 0$ and $\beta \neq 0$, then from the first of (4.40) we conclude that $a = d$. Since $a\delta + d\vartheta = 0$ and $a = d \neq 0$ we have $\delta + \vartheta = 0$. The relation (4.27) by virtue of $\delta + \vartheta = 0$ and first of (4.39) becomes $\beta b = 0$ and since $\beta \neq 0$ we have $b = 0$. From this we take $\xi = \zeta = r = \sigma = 0$.

From the above we conclude that in every case we have $\xi = \zeta = r = \sigma = 0$. Therefore the manifold $M = G/H$ with the canonical connection of the second kind is an affine symmetric space. Hence the *RR*-manifold (M, g, s) is a symmetric Riemannian manifold.

5. Let (M, g, s) be a simply connected four-dimensional *RR*-manifold whose regular *s*-structure $\{s_x\}$ has order greater than or equal to six. Therefore the linear transformation $S_0^{\mathbf{C}}$ on the vector space $T_0^{\mathbf{C}}(M) = T_0(M) \otimes \mathbf{C}$ has four distinct eigenvalues, where $T_0(M)$ is the tangent space of $M = G/H$ at its origin 0.

PROPOSITION (III). *Let (M, g, s) be a simply connected four-dimensional RR-manifold whose regular s-structure $\{s_x\}$ has order greater than five. Then the Riemannian manifold (M, g) is a symmetric space.*

PROOF. There exists a linear transformation S_0 on $T_0(M)$ from which we obtain the linear transformation $S_0^{\mathbf{C}}$ on $T_0^{\mathbf{C}}(M)$. Therefore there exists an orthonormal base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$ such that S_0 can be represented by the matrix.

$$S_0 = \begin{pmatrix} \cos\frac{2\pi v}{k} & -\sin\frac{2\pi v}{k} & 0 & 0 \\ \sin\frac{2\pi v}{k} & \cos\frac{2\pi v}{k} & 0 & 0 \\ 0 & 0 & \cos\frac{4\pi v}{k} & -\sin\frac{4\pi v}{k} \\ 0 & 0 & \sin\frac{4\pi v}{k} & \cos\frac{4\pi v}{k} \end{pmatrix}, \quad k \geqslant 6.$$

For the relations $A(S_0) = A(g_0) = 0$, which are valid for every $A \in L(H)$, we conclude that A can be represented as a matrix with respect to the $\{f_1, f_2, f_3, f_4\}$ by the following form

$$A = \begin{pmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & -p \\ 0 & 0 & p & 0 \end{pmatrix}. \tag{5.1}$$

If we set

$$V_1 = f_1 + if_2, \qquad V_2 = f_3 + if_4, \tag{5.2}$$

then V_1, V_2, $\overline{V}_1$, $\overline{V}_2$ form a base of $T_0^{\mathbf{C}}(M)$, which are the eigenvectors of $S_0^{\mathbf{C}}$ whose eigenvalues are $\varphi, \overline{\varphi}, \varphi^2, \overline{\varphi}^2$ respectively, where $\varphi = \cos(2\pi v/k) + i\sin(2\pi v/k)$. From Lemma (I) we obtain

$$\overline{T}_0(V_1, \overline{V}_1) = 0, \qquad \overline{T}_0(V_1, V_2) = 0, \tag{5.3}$$

$$\overline{T}_0(V_1, \overline{V}_2) = \lambda\overline{V}_1, \qquad \overline{T}_0(\overline{V}_1, V_2) = \overline{\lambda}V_1, \tag{5.4}$$

$$\overline{T}_0(\overline{V}_1, \overline{V}_2) = 0, \qquad \overline{T}_0(V_1, \overline{V}_2) = 0. \tag{5.5}$$

From the form of A, given by (5.1), we have

(5.6) $A(f_1) = -cf_2, \quad A(f_2) = cf_1, \quad A(f_3) = -pf_4, \quad A(f_4) = pf_3.$

From the relations (5.2) by virtue of (5.6) we obtain

(5.7) $A(V_1) = -ciV_1, \quad A(\bar{V}_1) = ci\bar{V}_1, \quad A(V_2) = -piV_2, \quad A(\bar{V}_2) = pi\bar{V}_2.$

Since $A(T_0) = 0$ and applying the derivation A to the first of (5.4) we take

$$\bar{T}_0(A(V_1), \bar{V}_2) + \bar{T}_0(V_1, A(\bar{V}_2)) = \lambda A(\bar{V}_1),$$

which by means of (5.7) and after same estimates implies $p = 2c$. Therefore the matrix (5.1) and the relations (5.6) take the form

(5.8)
$$A = c\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{pmatrix},$$

(5.9) $A(f_1) = -cf_2, \quad A(f_2) = cf_1, \quad A(f_3) = -2cf_4, \quad A(f_4) = 2cf_3.$

The relations (5.3), (5.4) and (5.5) by virtue of (5.2) and after some calculations give

(5.10) $\bar{T}_0(f_1, f_3) = \bar{T}_0(f_2, f_4) = ½(\zeta f_1 + \xi f_2),$

(5.11) $\bar{T}_0(f_2, f_3) = -\bar{T}_0(f_1, f_4) = ½(\zeta f_1 - \xi f_2),$

(5.12) $\bar{T}_0(f_1, f_2) = \bar{T}_0(f_3, f_4) = 0,$

where $\zeta = \mathrm{Re}(\lambda)$ and $\xi = \mathrm{Im}(\lambda)$.

Since $\dim(L(H)) \leqslant 1$, then the relations (4.9) and (4.10) by means of (5.6), (5.10), (5.11) and (5.12) imply

(5.13) $2[f_1, f_2] = \alpha e, \quad 2[f_1, f_3] = \zeta f_1 + \xi f_2 + \beta e,$

(5.14) $2[f_1, f_4] = -\xi f_1 + \zeta f_2 + \gamma e, \quad 2[f_2, f_3] = \xi f_1 - \zeta f_2 + \delta e,$

(5.15) $2[f_2, f_4] = \zeta f_1 + \xi f_2 + \vartheta e, \quad 2[f_3, f_4] = \nu e,$

(5.16) $[f_1, e] = -cf_2, \quad [f_2, e] = cf_1,$

(5.17) $[f_3, e] = -2cf_4, \quad [f_4, e] = 2cf_3,$

where

$$e = \begin{pmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & -2c \\ 0 & 0 & 2c & 0 \end{pmatrix}$$

for some fixed c.

These Lie brackets must satisfy the Jacobi identity from which we take

$$(5.18) \qquad c(2\gamma + \delta) = 0, \quad c(2\beta - \vartheta) = 0, \quad c(\gamma + 2\delta) = 0,$$

$$(5.19) \qquad c(\beta - 2\vartheta) = 0, \quad c\alpha = 0, \quad cv + \xi^2 + \zeta^2 = 0.$$

If $c = 0$, then the system of the equations (5.18) and (5.19) is satisfied when $\xi = \zeta = 0$ and hence the Riemannian manifold (M, g) is a symmetric space.

If $c \neq 0$ and $v \neq 0$, then from the system of equations (5.18) and (5.19) we have $\alpha = \beta = \gamma = \delta = \vartheta = 0$ and therefore the relations (5.13), (5.14), and (5.15) take the form

$$(5.20) \qquad [f_1, f_2] = 0, \quad 2[f_1, f_3] = \zeta f_1 + \xi f_2,$$

$$(5.21) \qquad 2[f_1, f_4] = -\xi f_1 + \zeta f_2 \quad 2[f_2, f_3] = \xi f_1 - \zeta f_2,$$

$$(5.22) \qquad 2[f_2, f_4] = \zeta f_1 + \xi f_2, \quad 2[f_3, f_4] = ve.$$

If $\dim(L(H)) = 1$, then $L(G)$ is a Lie algebra of five dimensions. Let $L(K)$ be an ideal of $L(G)$. Therefore we have $[\lambda, \mu] \in L(K)$ for every $\lambda \in L(K)$ and for every $\mu \in L(G)$. From the relations (5.16), (5.17), (5.20), (5.21) and (5.22) and after some estimates we conclude that $L(K) = \{0\}$. Therefore the Lie algebra $L(G)$ is simple. This is impossible because $\dim(L(G)) = 5$. Therefore the assumption $\dim(L(H)) = 1$ is false and hence $\dim(L(H)) = 0$. From this we conclude that $v = 0$ which implies $\xi = \zeta = 0$. This completes the proof of the proposition, i.e. (M, g) symmetric space.

6. We assume that RR-manifold (M, g, s) is simply connected and four dimensions whose regular s-structure $\{s_x\}$ has order five. Therefore the linear transformation $S_0^{\mathbf{C}}$ on $T_0^{\mathbf{C}}(M)$ has also four distinct eigenvalues. However this case is a little different from the previous one, because the set I contains all the eigenvalues of $S_0^{\mathbf{C}}$.

PROPOSITION (IV). *We consider a simply connected four-dimensional RR-manifold (M, g, s) whose regular s-structure $\{s_x\}$ has order five. Then the Riemannian manifold (M, g) is a symmetric space.*

PROOF. It is known that there exists an orthonormal base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$ such that the linear transformation S_0 on $T_0(M)$ can be represented by the matrix

$$S_0 = \begin{pmatrix} \cos\frac{2\pi}{5} & -\sin\frac{2\pi}{5} & 0 & 0 \\ \sin\frac{2\pi}{5} & \cos\frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \cos\frac{4\pi}{5} & -\sin\frac{4\pi}{5} \\ 0 & 0 & \sin\frac{4\pi}{5} & \cos\frac{4\pi}{5} \end{pmatrix}.$$

Since we have $A(S_0) = A(g_0) = 0$ for every $A \in L(H)$ we conclude that A can be represented by the matrix

$$A = \begin{pmatrix} 0 & -b & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & -d \\ 0 & 0 & d & 0 \end{pmatrix} \tag{6.1}$$

with respect to the basis $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$.

From the form of the matrix A we have

$$A(f_1) = bf_2, \quad A(f_2) = -bf_1, \quad A(f_3) = df_4, \quad A(f_4) = -df_3. \tag{6.2}$$

If we put

$$V_1 = f_1 + if_2, \qquad V_2 = f_3 + if_4, \tag{6.3}$$

then by means of (6.2) and after some estimates we obtain

$$A(V_1) = -ibV_1, \quad A(\overline{V}_1) = ib\overline{V}_1, \quad A(V_2) = idV_2, \quad A(\overline{V}_2) = id\overline{V}_2. \tag{6.4}$$

From the form of the eigenvalues of S_0^C and by means of Lemma (I) we have

$$\overline{T}_0(V_1, \overline{V}_1) = 0, \qquad \overline{T}_0(V_1, V_2) = \lambda\overline{V}_1, \qquad \overline{T}_0(V_1, \overline{V}_2) = \mu\overline{V}_1, \tag{6.5}$$

$$\overline{T}_0(V_2, \overline{V}_2) = 0, \qquad \overline{T}_0(\overline{V}_1, V_2) = \overline{\mu}V_1, \qquad \overline{T}_0(\overline{V}_1, \overline{V}_2) = \overline{\lambda}V_2, \tag{6.6}$$

which by virtue of (6.3) and after some calculations imply

$$\overline{T}_0(f_1, f_2) = \overline{T}_0(f_3, f_4) = 0, \tag{6.7}$$

(6.8) $$\bar{T}_0(f_1, f_3) = \tfrac{1}{2}(rf_1 + \sigma f_2 + \zeta f_3 + \xi f_4),$$

(6.9) $$\bar{T}_0(f_1, f_4) = \tfrac{1}{2}(-\sigma f_1 + rf_2 + \xi f_3 - \zeta f_4),$$

(6.10) $$\bar{T}_0(f_2, f_3) = \tfrac{1}{2}(\sigma f_1 - rf_2 + \xi f_3 - \zeta f_4),$$

(6.11) $$\bar{T}_0(f_2, f_4) = \tfrac{1}{2}(rf_1 + \sigma f_2 - \zeta f_3 - \xi f_4),$$

where $r = \mathrm{Re}(\mu)$, $\sigma = \mathrm{Im}(\mu)$, $\zeta = \mathrm{Re}(\lambda)$ and $\xi = \mathrm{Im}(\lambda)$.

Since $A(T_0) = 0$ for every $A \in L(H)$ and applying the derivations A to the second and third of (6.5) we take

$$\bar{T}_0(A(V_1), V_2) + \bar{T}_0(V_1, A(V_2)) = \lambda A(\bar{V}_2),$$

$$\bar{T}_0(A(V_1), \bar{V}_2) + \bar{T}_0(V_1, A(\bar{V}_2)) = \mu A(\bar{V}_1),$$

which by means of (6.4) and after some estimates give $\lambda(2d + b) = 0$, $\mu(2b + d) = 0$ from which we have $\lambda = \mu = 0$ or $\lambda = 0$, $\mu \neq 0$ and $d = -2b$ or $\lambda \neq 0$, $\mu = 0$ and $b = -2d$ or finally $\lambda \neq 0$, $\mu \neq 0$ and therefore $d = b = 0$.

If $\lambda = \mu = 0$, then the Riemannian manifold (M, g) is a symmetric space. If $\lambda = 0$ and $\mu \neq 0$, then we are in the case of §5 and hence (M, g) is a symmetric space.

Now we assume that $\lambda \neq 0$, $\mu = 0$ and $b = -2d$ then (6.2), (6.8), (6.9), (6.10) and (6.11) take the form

(6.12) $$A(f_1) = -2df_2, \quad A(f_2) = 2df, \quad A(f_3) = -df_4, \quad A(f_4) = -df_3,$$

(6.13) $$\bar{T}_0(f_1, f_3) = \tfrac{1}{2}(\zeta f_3 + \xi f_4), \qquad \bar{T}_0(f_1, f_4) = \tfrac{1}{2}(\xi f_3 - \zeta f_4),$$

(6.14) $$\bar{T}_0(f_2, f_3) = \tfrac{1}{2}(\xi f_3 - \zeta f_4), \qquad \bar{T}_0(f_2, f_4) = -\tfrac{1}{2}(\zeta f_3 + \xi f_4).$$

Since $\dim(L(H)) \leqslant 1$, then from (4.9), (4.10), (6.7), (6.8), (6.9), (6.10) and (6.11) we obtain

$$[f_1, f_2] = \alpha e, \quad [f_1, f_3] = -\tfrac{1}{2}(\zeta f_3 + \xi f_4) + \beta e,$$

$$[f_1, f_4] = -\tfrac{1}{2}(\xi f_3 - \zeta f_4) + \gamma e, \quad [f_2, f_3] = -\tfrac{1}{2}(\xi f_3 - \zeta f_4) + \delta e,$$

$$[f_2, f_4] = \tfrac{1}{2}(\xi f_3 + \zeta f_4) + \vartheta e, \quad [f_3, f_4] = \nu e,$$

$$[f_1, e] = 2df_2, \quad [f_2, e] = -2df_1, \quad [f_3, e] = -df_4, \quad [f_4, e] = df_3,$$

where e is a vector of $L(H)$.

If we use the same arguments as in §5 we obtain in this case that, the Riemannian manifold (M, g) is a symmetric space.

Finally, if $\lambda \neq 0$ and $\mu \neq 0$, then $b = d = 0$ and hence $\bar{R}_0 = 0$. Therefore the relation (4.9) becomes $[X, Y] = -\bar{T}_0(X, Y)$ which by means of (6.5) and (6.6) implies

(6.15) $[V_1, \bar{V}_1] = 0, \quad [V_1, V_2] = \lambda \bar{V}_2, \quad [V_1, \bar{V}_2] = \mu \bar{V}_1,$

(6.16) $[V_2, \bar{V}_2] = 0, \quad [\bar{V}_1, \bar{V}_2] = \bar{\lambda} V_2, \quad [\bar{V}_1, V_2] = \bar{\mu} V_1.$

These Lie brackets must satisfy the Jacobi identity. Therefore we have

$$[[V_1, \bar{V}_1], V_2] + [[V_2, V_1], \bar{V}_1] + [[\bar{V}_1, V_2], V_1] = 0$$

which by virtue of (6.15) and (6.16) and after some estimates takes the form $\lambda\bar{\lambda}V_2 = 0$ or $\lambda = 0$ similarly we have

$$[[V_2, \bar{V}_2], \bar{V}_1] + [[\bar{V}_1, V_1], \bar{V}_2] + [\bar{V}_2, \bar{V}_1], V_1] = 0$$

which by means of (6.15) and (6,16) and after some calculations we obtain $\mu\bar{\mu}V_1 = 0$ or $\mu = 0$.

This completes the proof of Proposition (IV).

7. It has been proved that a simply connected four-dimensional *RR*-manifold (M, g, s) whose regular s-structure $\{s_x\}$ has order different than four is a symmetric space. Now we assume that the regular s-structure $\{s_x\}$ has order four and therefore the linear transformation $S_0^{\mathbf{C}}$ on the vector space $T_0^{\mathbf{C}}(M)$ has three distinct eigenvalues which are -1, i and $-i$. The eigenvalue -1 is double.

We also have a linear transformation S_0 on the tangent space $T_0(M)$ of $M = G/H$ at its origin 0. Hence there exists an orthonormal base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$ such that the linear transformation S_0 can be represented by the matrix

$$S_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

PROPOSITION (V). *Let (M, g, s) be a simply connected four-dimensional RR-manifold whose regular s-structure $\{s_x\}$ has order four. Then there are four vector fields X, Y, Z, W on M satisfying the relations*

(i)
$$\bar{T}(X, Z) = -X, \quad \bar{T}(Y, Z) = Y, \quad \bar{T}(X, Y) = 0,$$
$$\bar{T}(X, W) = -X, \quad \bar{T}(Y, W) = Y, \quad \bar{T}(W, Z) = 0,$$

(ii) $$g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = \frac{1}{|\gamma|^2}, \quad g(W, W) = \frac{1}{|\delta|^2}.$$

PROOF. From the form of the linear transformation S_0 we have

(7.1) $S_0(f_1) = -f_2, \quad S_0(f_2) = f_1, \quad S_0(f_3) = -f_3, \quad S_0(f_4) = -f_4.$

If we set

$$V_1 = f_1 + if_2, \quad U_2 = f_3, \quad W_2 = f_4 \tag{7.2}$$

then by means of (7.1) we obtain

$$S_0(V_1) = -iV_1, \quad S_0(U_2) = -U_2, \quad S_0(W_2) = -W_2. \tag{7.3}$$

From Lemma (I) and the form of the eigenvalues of $S_0^{\mathbf{C}}$ we have

$$\bar{T}_0(V_1, V_1) = 0, \quad \bar{T}_0(\bar{V}_1, \bar{V}_1) = 0, \quad \bar{T}_0(U_2, W_2) = 0, \tag{7.4}$$

$$\bar{T}_0(V_1, \bar{V}_1) = 0, \quad \bar{T}_0(V_1, U_2) = \gamma \bar{V}_1, \quad \bar{T}_0(V_1, W_2) = \delta \bar{V}_1, \tag{7.5}$$

$$\bar{T}_0(\bar{V}_1, U_2) = \bar{\gamma} V_1, \quad \bar{T}_0(V_1, W_2) = \bar{\delta} V_1. \tag{7.6}$$

The Lie algebra $L(H)$ consists of linear transformations A on $T_0(M)$ which satisfy $A(S_0) = A(g_0) = A(\bar{R}_0) = A(\bar{T}_0) = 0$.

From the relations $A(S_0) = A(g_0) = 0$ we conclude that A can be represented by the matrix

$$A = \begin{pmatrix} 0 & b & 0 & 0 \\ -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with respect to the base $\{f_1, f_2, f_3, f_4\}$ of $T_0(M)$.

From the form of linear transformation A we obtain

$$A(f_1) = -bf_2, \quad A(f_2) = bf_1, \quad A(f_3) = 0, \quad A(f_4) = 0. \tag{7.7}$$

From (7.2) by virtue of (7.7) we conclude that

$$A(V_1) = ibV_1, \quad A(\bar{V}_1) = -ib\bar{V}_1, \quad A(U_2) = 0, \quad A(W_2) = 0. \tag{7.8}$$

Since $A(\bar{T}_0) = 0$, then from the second of (7.5) we have

$$\bar{T}_0(A(V_1), U_2) + \bar{T}_0(V_1, A(V_2)) = A(V_1)$$

which by means of (7.8) becomes $bi\bar{T}_0(V_1, U_2) = -\gamma bi\bar{V}_1$ which implies $b = 0$. Therefore $A = 0$ and $L(H) = \{0\}$.

The complexification of the tangent space $T_0(M)$ gives the four-dimensional complex vector space $T_0^{\mathbf{C}}(M)$ which can be written $T_0^{\mathbf{C}}(M) = L^{\mathbf{C}} \oplus N^{\mathbf{C}}$, where $L^{\mathbf{C}}$ is the vector subspace of $T_0^{\mathbf{C}}(M)$ which is spanned by the vectors V_1 and $\bar{V}_1$ and $N^{\mathbf{C}}$ the complexification of the vector space which is the eigenspace of S_0 with eigenvalue -1.

On the vector space L^C we consider a linear transformation B_{U_2} defined by

$$B_{U_2}: V_1 \longrightarrow \bar{T}_0(V_1, U_2) = \gamma \bar{V}_1, \quad B_{U_2}: \bar{V}_1 \longrightarrow \bar{T}_0(\bar{V}_1, U_2) = \bar{\gamma} V_1$$

which has the property such that $B_{U_2}^2: V_1 \longrightarrow \gamma\bar{\gamma} V_1 = |\gamma|^2 V_1$.

If we set $U_2/|\gamma| = Z$, then we have another linear transformation B_Z on L^C with the property $B_z^2 = \text{id}$, whose eigenvalues are -1 and $+1$. Let X, Y be the corresponding real unit eigenvectors determined uniquely up to sign.

Similarly from the third of (7.5) we obtain another linear transformation B_W on L^C from which we construct a new linear transformation B_W on L^C such that $B_W^2 = \text{id}$. Therefore the eigenvalues of B_W are 1 and -1. Hence B_W has the same real unit eigenvectors X, Y.

Since $\bar{R} = 0$ and M simply connected, we can extend the vectors X, Y, Z, W to parallel vector fields on M, which will be denoted by the same symbols. From the properties of the linear transformations B_Z and B_W we obtain

(7.9) $\bar{T}(X, Z) = -X, \quad \bar{T}(Y, Z) = Y, \quad \bar{T}(X, W) = -X, \quad \bar{T}(Y, W) = Y.$

The vectors X, Y form a basis of the vector space L. Therefore the vectors V_1, $\bar{V}_1$ can be written

$$V_1 = \alpha_1 X + \beta_1 Y + i(\alpha_2 X + \beta_2 Y), \qquad \bar{V}_1 = \alpha_1 X + \beta_1 Y - i(\alpha_2 X + \beta_2 Y),$$

where

$$\alpha_1 X + \beta_1 Y = f_1, \qquad \alpha_2 X + \beta_2 Y = f_2. \tag{7.10}$$

The first of (7.4) by means of the first of (7.2) can be written $\bar{T}_0(f_1 + if_2, f_1 - if_2) = 0$ from which we have $\bar{T}_0(f_1, f_2) = 0$, that by virtue of (7.10) becomes

$$(\alpha_1\beta_2 - \beta_1\alpha_2)\bar{T}_0(X, Y) = 0. \tag{7.11}$$

The third of the relations (7.4) can be written

$$\bar{T}_0(U_2/|\gamma|, W_2/|\delta|) = \bar{T}_0(Z, W) = 0. \tag{7.12}$$

From (7.11) and (7.12) we have

$$\bar{T}(X, Y) = \bar{T}(Z, W) = 0. \tag{7.13}$$

(ii) The vectors X, Y, Z, W form an orthonormal base of the tangent space $T_0(M)$ with respect to the inner product on it defined by g_0. Therefore we obtain

$$g_0(X, X) = g_0(Y, Y) = 1, \quad g_0(U_2, U_2) = g_0(W_2, W_2) = 1. \tag{7.14}$$

The last two relations of (7.14) by means of $U_2/|\delta| = Z$ and $W_2/|\delta| = W$ become

$$g_0(Z, Z) = 1/|\gamma|^2, \quad g_0(W, W) = 1/|\delta|^2. \tag{7.15}$$

Therefore (7.14) and (7.15) for the vector fields X, Y, Z, W take the form

$$g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = 1/|\gamma|^2, \quad g(W, W) = 1/|\delta|^2.$$

PROPOSITION (VI). *Let (M, g, s) be a simply connected four-dimensional RR-manifold whose regular s-structure $\{s_x : x \in M\}$ has order 4. Then there are four vector fields X, Y, Z, W satisfying*

$$[X, Z] = [X, W] = X, \quad [Y, Z] = [Y, W] = -Y, \quad [X, Y] = [W, Z] = 0,$$

$$SX = \epsilon Y, \quad SY = \epsilon' X, \quad SZ = -Z, \quad SW = -W,$$

where $\epsilon, \epsilon' = \pm 1$ and S is the symmetry tensor field.

PROOF. It is known that the following formula holds

$$\bar{T}(X_1, Y_1) = \bar{\nabla}_{X_1} Y_1 - \bar{\nabla}_{Y_1} X_1 - [X_1, Y_1]. \tag{7.16}$$

Since $\bar{R} = 0$, $\bar{\nabla}_{X_1} Y_1 = \bar{\nabla}_{Y_1} X_1 = 0$. Therefore from (7.16) we obtain

$$\bar{T}(X, Y) = -[X, Y]. \tag{7.17}$$

The relations (7.9) and (7.13) by virtue of (7.17) imply

$$[X, Z] = X, \quad [Y, Z] = -Y, \quad [X, Y] = 0, \tag{7.18}$$

$$[X, W] = X, \quad [Y, W] = -Y, \quad [W, Z] = 0. \tag{7.19}$$

The symmetry tensor field S of $\{s_x\}$ has order 4 and for each $P \in M$, S_P is an orthogonal transformation on $T_P(M)$ which satisfies the relations

$$S_P(Z_P) = -Z_P, \quad S_P(W_P) = -W_P, \qquad \forall P \in M$$

from which we obtain

$$SZ = -Z, \quad SW = -W. \tag{7.20}$$

It is known that the tensor field S preserves the tensor field $\bar{T}$. Therefore from the first and the second of (7.9) we obtain

$$S(\bar{T}(X, Z)) = \bar{T}(SX, SZ) = -SX, \qquad S(\bar{T}(Y, Z)) = \bar{T}(SY, SZ) = SY,$$

which by means of (7.20) take the form

$$\bar{T}(SX, Z) = SX, \qquad \bar{T}(SY, Z) = -SY. \tag{7.21}$$

Similarly from the third and the fourth of (7.9) by virtue of (7.20) we conclude that

(7.22) $$\bar{T}(SX, W) = SX, \qquad \bar{T}(SY, W) = -SY.$$

From the relations (7.9) by means of (7.21) and (7.22) we have

(7.23) $$SX = \epsilon Y, \qquad SY = \epsilon' X,$$

where $\epsilon, \epsilon' = \pm 1$.

THEOREM (IV). *Let (M, g, s) be a simply connected four-dimensional RR-manifold whose regular s-structure $\{s_x : x \in M\}$ has order four. Then for the manifold (M, g) we have $M = R^4(x_1, x_2, x_3, x_4)$ provided with the Riemannian metric*

$$ds^2 = e^{2(x_3+x_4)}dx_1^2 + e^{-2(x_3+x_4)}dx_2^2 + \frac{1}{\lambda^2}dx_3^2 + \frac{1}{\mu^2}dx_4^2,$$

λ, μ *constants* $\neq 0$.

It can also be represented as a Lie group $G = G_1 \times R_*$ where G_1 is isomorphic to the Lie group of all hyperbolic motions of an oriented affine plane and $R_* = R - \{0\}$ provided with a special left-invariant Riemannian metric.

PROOF. Since $L(H) = \{0\}$, we conclude that the Lie algebra $L(G)$ of G is four dimensions whose Lie bracket satisfies the relations (7.18) and (7.19). The adjoint group $\mathrm{Int}(L(G))$ of $L(G)$ is generated by the elements of the form e^{2dX_1}, $X_1 \in L(G)$ [3, p. 117].

The group $\mathrm{Int}(L(G))$ can be identified with the group of all matrices of the form

$$W = \begin{pmatrix} e^{-\gamma} & 0 & \lambda & 0 \\ 0 & e^{\gamma} & \mu & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we obtain as a Lie group G_1 the set of all matrices of the form W, then G_1 is isomorphic to the Lie group G_2 consisting of matrices of the form

$$Y = \begin{pmatrix} e^{-\gamma} & 0 & \lambda \\ 0 & e^{\gamma} & \mu \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie group G_2 is isomorphic to the Lie group of all hyperbolic motions of an oriented affine plane:

$$u' = e^{-\gamma}u + \lambda, \qquad v' = e^{\gamma}v + \mu.$$

From the above and [3, p. 119] we conclude that the Lie group $G = G_1 \times R_*$. From the form of the Lie group G, $\dim G = 4$ and the results given in [4, p. 362] we conclude that G is diffeomorphic to $R^4(x_1, x_2, x_3, x_4)$.

The Lie algebra $L(G)$ of left vector fields on G has a base consisting of the following vector fields:

(7.25) $$X = e^{-(x_3+x_4)}\frac{\partial}{\partial x_1}, \quad Y = e^{(x_3+x_4)}\frac{\partial}{\partial x_2}, \quad Z = \frac{\partial}{\partial x_3}, \quad W = \frac{\partial}{\partial x_4}.$$

These vector fields satisfy the equations

(7.26) $$[X, Z] = X, \quad [Y, Z] = -Y, \quad [X, Y] = 0,$$

(7.27) $$[X, W] = X, \quad [Y, W] = -Y, \quad [Z, W] = 0.$$

We define a left invariant Riemannian metric g on G by the conditions

(7.28) $$g(X, Y) = g(X, Z) = g(X, W) = g(Y, Z) = g(Y, W) = g(Z, W) = 0,$$

(7.29) $$g(X, X) = g(Y, Y) = 1, \quad g(Z, Z) = \frac{1}{\lambda^2}, \quad g(W, W) = \frac{1}{\mu^2}, \qquad \lambda, \mu \neq 0.$$

The metric g with respect to the coordinate system (x_1, x_2, x_3, x_4)

(7.30) $$ds^2 = e^{2(x_3+x_4)}dx_1^2 + e^{-2(x_3+x_4)}dx_2^2 + \frac{1}{\lambda^2}dx_3^2 + \frac{1}{\mu^2}dx_4^2.$$

At each point P of the manifold M we have a linear transformation S_P on $T_P(M)$ which is defined by

$$S_P(X_P) = \epsilon Y_P, \quad S_P(Y_P) = \epsilon X_P, \quad S_P(Z_P) = -Z_P, \quad S_P(W_P) = -W_P.$$

The linear transformation S_P defines a regular s-structure $\{s_x\}$ on the manifold M whose order is four.

On the manifold $M = G$ we define the canonical (–)-connection $\overline{\nabla}$ for which we have

(7.31) $$\overline{T}(U, V) = -[U, V], \quad \overline{\nabla}\overline{T} = \overline{R} = 0, \qquad \forall U, V \in L(G).$$

The vector fields X, Y, Z, W are parallel with respect to the connection $\overline{\nabla}$.

From the relations (7.21), (7.22), (7.28) and (7.29) by means of (7.20) and (7.23) we obtain that $\overline{T}$ and g are invariants by S and an addition S and g are parallel with respect to the connection $\overline{\nabla}$.

From the known theorem [6] we obtain that S gives rise to a nonparallel regular s-structure $\{s_x\}$ of order four on the Riemannian manifold (G, g) and since the manifold M is simply connected we conclude that, this is isometric to the manifold (G, g).

From the fact that the vector fields X, Y, Z, W are parallel with respect to the connection $\bar{\nabla} = \nabla - D$, then we conclude that

$$\nabla_V(U) = D(V, U), \qquad V, U \in \{X, Y, Z, W\}.$$

From this relation and the formula

$$2g(D(V, U'), V') = g(\bar{T}(V, U), V') + g(\bar{T}(V, V'), U) + g(\bar{T}(U, V'), V),$$

$U, V, V' \in \{X, Y, Z, W\}$ and after some calculations we obtain

$$(7.32)\qquad \begin{aligned} &\nabla_X X = -\lambda^2 Z - \mu^2 W, \quad \nabla_X Y = 0, \quad \nabla_X Z = \nabla_X W = X, \\ &\nabla_Y X = 0, \quad \nabla_Y Y = \lambda^2 Z + \mu^2 W, \quad \nabla_Y Z = \nabla_Y W = -Y, \\ &\nabla_Z X = \nabla_Z Y = \nabla_Z Z = \nabla_Z W = 0, \\ &\nabla_W X = \nabla_W Y = \nabla_W Z = \nabla_W W = 0. \end{aligned}$$

Therefore the Gauss curvature in the basic 2-directions are given by $\sigma(X, Y) = \lambda^2 + \mu^2$, $\sigma(V, U) = 0$, where $V, U \in \{X, Y, Z, W\}$ but it is not simultaneously $U = X$ and $V = Y$ or $U = Y$ and $V = X$. It is easily seen that the only tangent 2-planes with sectional curvature $\lambda^2 + \mu^2$ are those in the distribution spanned by $\{X, Y\}$.

Therefore this family must be preserved by any isometry I of (M, g). From this we obtain

$$I_*Z = \epsilon_1 Z, \;\; I_*W = \epsilon_2 W, \;\; I_*X = X\cos\vartheta + Y\sin\vartheta, \;\; I_*Y = \epsilon(-X\sin\vartheta + Y\cos\vartheta),$$

where ϵ_1, ϵ_2 and $\epsilon = \pm 1$ and the parameter ϑ is a real function on M.

Also each isometry I of (M, g) is an affine transformation, i.e. $\nabla_{I_*U} I_*V = I_*(\nabla_U V)$ for any two vector fields U, V on M. By examining the cases $(U = X, V = Z)$ $(U = X, V = W)$, $(U = Y, V = Z)$ and $(U = Y, V = W)$ and by means of (7.32) and after some calculations we obtain

$$\begin{aligned} &I_*X = \delta X, \quad I_*Y = \delta' Y, \quad I_*Z = Z, \quad I_*W = W, \\ &I_*X = \delta Y, \quad I_*Y = \delta' X, \quad I_*Z = -Z, \quad I_*W = -W, \end{aligned}$$

where $\delta, \delta' = \pm 1$.

The isotropy subgroup of G at any point $P \in M$ is finite of order 8 and contains exactly four symmetries. The manifold (M, g) with this metric is not symmetric.

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